Since for a given circle and a given exterior point the product of any secant through the point and the exterior segment of the secant is equal to the square of the tangent from the point to the circle we have $A'B \cdot AB = x_2^2 + y_2^2 - k^2$ and $A'B \cdot A'B' = (x_3 - k')^2 + y_3^2 - k^2$. Hence,

$$AB + A'B' = \frac{x_2^2 + y_2^2 - k^2 + (x_3 - k')^2 + y_3^2 - k^2}{A'B}.$$

By use of $A'B = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$ and the relations given above this equation readily reduces to $AB + A'B' = 2k'/\sqrt{1 + m^2}$, which is independent of k and dependent upon k' and θ only, as was to be shown.

Also solved by T. M. BLAKSLEE, PAUL CAPRON, OLIVE C. HAZLETT, C. N. MILLS, E. J. OGLESBY, ARTHUR PELLETIER, C. P. SOUSLEY, C. E. STROMQUIST, and H. S. UHLER.

2804 [1920, 32]. Proposed by T. H. GRONWALL, Washington, D. C.

Show that for $|x| < 1$

$$\frac{1}{\sqrt{1 - x^4}} \int_0^x \frac{dx}{\sqrt{1 - x^4}} = x + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} x^{4n+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1 - x^4}} \right)^2 = x^2 + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} \frac{x^{4n+2}}{2n + 1}.$$

SOLUTION BY O. S. ADAMS, Coast and Geodetic Survey.

The general forms for such integrals may be derived and the formulas in the statement will then be special cases of the derived forms. We then state the problem as follows:

For $|x| < 1$, s a positive integer,

$$\frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = x + \sum_1^\infty \frac{\frac{s+2}{2} \cdot \frac{3s+2}{2} \cdots \frac{(2n-1)s+2}{2}}{(s + 1)(2s + 1) \cdots (ns + 1)} x^{ns+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1 - x^s}} \right)^2 = x^2 + \sum_1^\infty \frac{\frac{s+2}{2} \cdot \frac{3s+2}{2} \cdots \frac{(2n-1)s+2}{2}}{(s + 1)(2s + 1) \cdots (ns + 1)} \frac{2x^{ns+2}}{ns + 2}.$$

Let

$$\phi(x) = \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{d}{dx} \phi(x) = \frac{1}{\sqrt{1 - x^s}},$$

and

$$2\phi(x) \frac{d}{dx} \phi(x) = \frac{2}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}}.$$

By integration,

$$\phi^2(x) = 2 \int_0^x \left[\frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} \right] dx.$$

Therefore, if the first series is multiplied by $2dx$ and integrated from 0 to x, the second series will be obtained.

Let

$$y = \frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{dy}{dx} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}}{(1 - x^s)^{3/2}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}y}{1 - x^s};$$

or

$$(1 - x^s)\frac{dy}{dx} - \frac{s}{2}x^{s-1}y - 1 = 0.$$

The function $1/\sqrt{1 - x^s}$ can be developed as a power series with unit radius of convergence, this can be integrated term by term, and the result multiplied by the original series, giving for y an expression of the form $y = x + ax^{s+1} + bx^{2s+1} + cx^{3s+1} + dx^{4s+1} + \cdots$. Now the series is absolutely convergent and we can determine the coefficients a, b, c, by substituting this series in the differential equation. We obtain

$$\begin{aligned} 1 + (s + 1)ax^s + (2s + 1)bx^{2s} + (3s + 1)cx^{3s} + (4s + 1)dx^{4s} + \cdots \\ - x^s - (s + 1)ax^{2s} - (2s + 1)bx^{3s} - (3s + 1)cx^{4s} - \cdots \\ - 1 - \frac{s}{2}x^s - \frac{s}{2}ax^{2s} - \frac{s}{2}bx^{3s} - \frac{s}{2}cx^{4s} - \cdots \equiv 0; \end{aligned}$$

so that the coefficients of the various powers of x must be identically zero. We must then have

$$a = \frac{\frac{s+2}{2}}{s+1}, \qquad b = \frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}}{(s+1)(2s+1)},$$

and

$$c = \frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdot\frac{5s+2}{2}}{(s+1)(2s+1)(3s+1)}.$$

The $(n + 1)$th coefficient thus becomes equal to the expression

$$\frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdots\frac{(2n-1)s+2}{2}}{(s+1)(2s+1)\cdots(ns+1)}.$$

This proves the formula

$$\frac{1}{\sqrt{1-x^s}}\int_0^x \frac{dx}{\sqrt{1-x^s}} = x + \sum_1^\infty \frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdots\frac{(2n-1)s+2}{2}}{(s+1)(2s+1)\cdots(ns+1)}x^{ns+1}.$$

By multiplying this series by $2dx$ and integrating from 0 to x, we obtain the second formula

$$\left(\int_0^x \frac{dx}{\sqrt{1-x^s}}\right)^2 = x^2 + \sum_1^\infty \frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdots\frac{(2n-1)s+2}{2}}{(s+1)(2s+1)\cdots(ns+1)}\frac{2x^{ns+2}}{ns+2}.$$

These formulas can easily be verified for $s = 1$ and they can be compared with the known forms for $s = 2$. When $s = 4$, they become identical with those stated in the problem.

$$\frac{1}{\sqrt{1-x^4}}\int_0^x \frac{dx}{\sqrt{1-x^4}} = x + \sum_1^\infty \frac{3\cdot 7\cdots(4n-5)(4n-1)}{5\cdot 9\cdots(4n-3)(4n+1)}x^{4n+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1-x^4}}\right)^2 = x^2 + \sum_1^\infty \frac{3\cdot 7\cdots(4n-5)(4n-1)}{5\cdot 9\cdots(4n-3)(4n+1)}\frac{x^{4n+2}}{2n+1}.$$

The last formula is given by Gauss, *Werke*, vol. 3, p. 406.

Also solved by ELIJAH SWIFT and the Proposer.